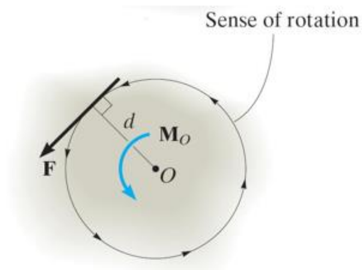
Module 4

Force System Resultants

Sections 4.1-4.4 Introduction to Moments

A moment is a measure of the tendency of a body to rotate (also called a torque). This tendency to rotate occurs at a point that is not on the line of action of force that is applied to a body.



In a 2D plane, the moment about a point O is found by:

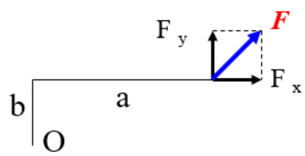
$$M_O = Fd$$

where d is the moment arm (or perpendicular distance) from the axis O to the line of action of the force. The "direction" of rotation is called the sense of rotation and is either clockwise or counterclockwise.

The typical sign convention is:

- Positive – Counterclockwise
- Negative – Clockwise

If the perpendicular distance to the line of action of a force is not easy to find, the force can be divided into components (this is called the principle of moments):



In the case of the figure above, the moment about O would be computed $M_O = F_y a - F_x b$

In 3D space, the moment about a point can be found by taking the **cross product** of the force vector and the position vector. The physical definition of the cross product is:

$$\vec{M}_O = \vec{r} \times \vec{F} = rF \sin\theta\, \hat{n}$$

where θ is the angle between the position vector $\vec{r}$ and the force vector $\vec{F}$ (measured "counterclockwise from the position to the force), and $\hat{n}$ is a unit vector normal to both the position and force vectors. The direction of $\hat{n}$ is determined by the "right-hand rule".

Mathematically, the cross product can be found by taking the determinant of the two vectors:

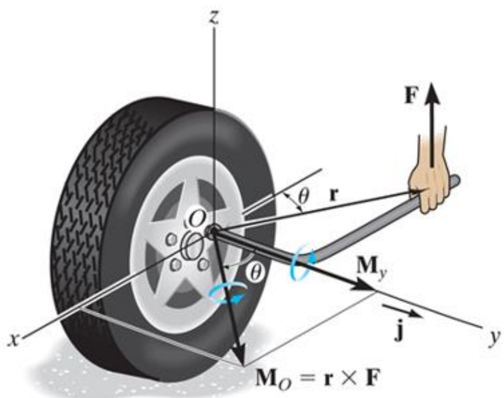
$$\vec{M}_O = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} = \hat{\imath} \begin{vmatrix} r_y & r_z \\ F_y & F_z \end{vmatrix} - \hat{\jmath} \begin{vmatrix} r_x & r_z \\ F_x & F_z \end{vmatrix} + \hat{k} \begin{vmatrix} r_x & r_y \\ F_x & F_y \end{vmatrix}$$

$$\vec{M}_O = \left(r_y F_z - r_z F_y\right)\hat{\imath} - \left(r_x F_z - r_z F_x\right)\hat{\jmath} + \left(r_x F_y - r_y F_x\right)\hat{k}$$

Note that the answer is another vector, with x-, y-, and z-components. These components refer to the tendency of rotation about the x-, y-, and z-axes, respectively.

Section 4.5 Moment of a Force about a Specified Axis

Sometimes, a force can cause a moment about an axis of rotation, but we may need to find the tendency of rotation about a different axis. For example:



In this example, the force creates a moment about an axis in the xy-plane. But in order to determine the rotation that will actually unscrew the lug, the rotation about the y-axis is required.

The steps to find the moment about a specified axis "a" are as follows:

- Find the unit vector $\hat{u}_a$ about the specified axis
- Find the moment $\vec{M}_O$ about any arbitrary point in the axis.
- Find the component of $\vec{M}_O$ along the a-axis using the dot product:

$$M_a = \hat{u}_a \cdot \vec{M}_O$$

Note that following the cross-product definition of moment, the equation above can also be written as:

$$M_a = \hat{u}_a \cdot \vec{M}_O = \hat{u}_a \cdot \left(\vec{r} \times \vec{F}\right) = \begin{vmatrix} u_{a_x} & u_{a_y} & u_{a_z} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

This is called the "scalar triple product".

Try it yourself: Problem F4-15 from the book.

Section 4.6 Moment of a Couple

A couple consists of two parallel forces that have the same magnitude but opposite directions, separated by a perpendicular distance d (or $\vec{r}$ in vector form).

A couple moment is the moment produced by a couple, and it is found by finding the sum of moments of the couple forces about any arbitrary point.

The couple moment is found as follows, in scalar and vector forms, respectively:

$$M_O = Fd$$

$$\vec{M}_O = \vec{r} \times \vec{F}$$

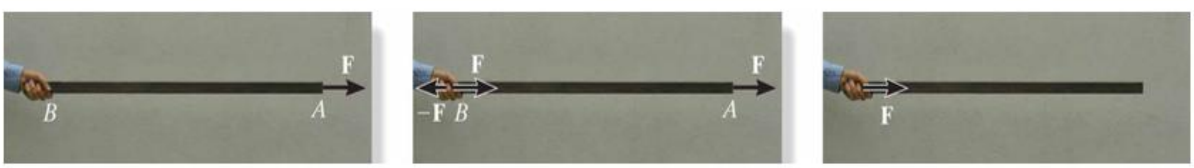
Couple moments can be added using standard vector addition rules.

Try it yourself: Problem F4-21 & Problem F4-24

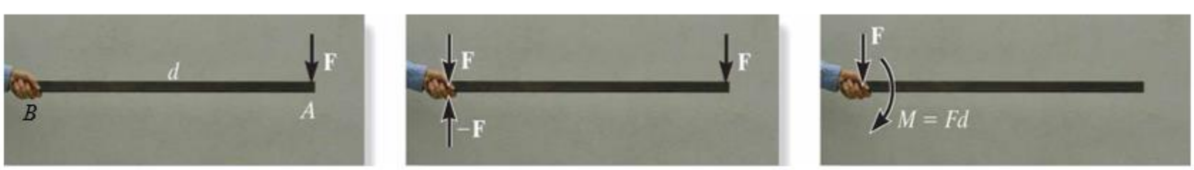
Section 4.7 Simplification of a Force and Couple System

When a series of forces and couple moments are acting on a body, it is convenient to reduce them to a simpler form with an equivalent system, which consists of a single resultant force and a single resultant moment.

Example, find an equivalent system of forces acting on B.



Since A and B are on the same line of action of the force, finding the equivalent force is the same as "sliding" the vector.



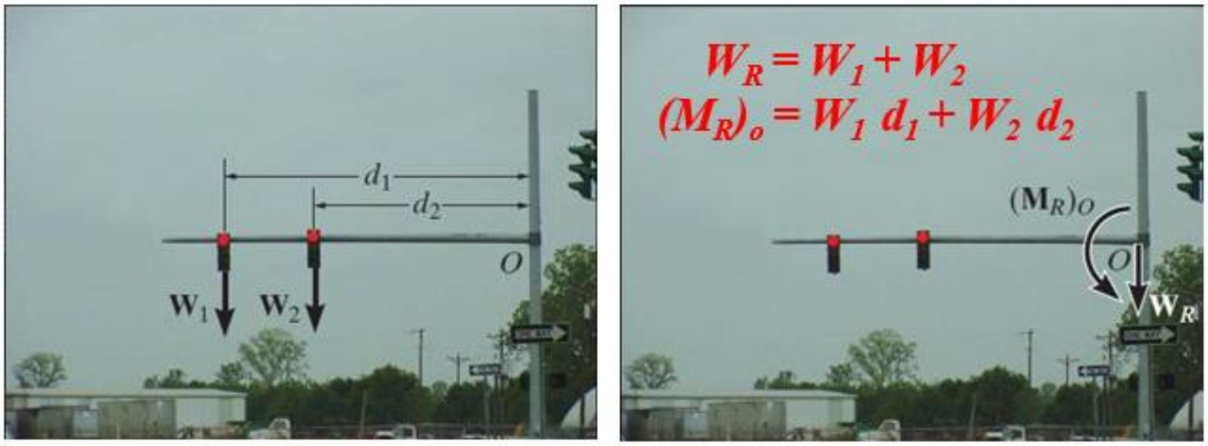
If A and B do not lie on the same line of action of the force, the result will be an equivalent force and equivalent couple moment.

The resultant force and resultant couple moment acting at a point O can be found by the following equations:

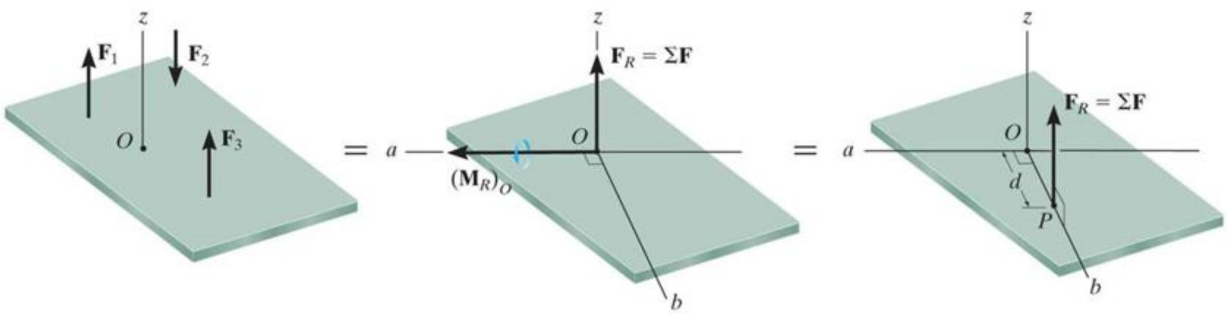
$$\vec{F}_R = \sum \vec{F}$$

$$\left(\vec{M}_R\right)_O = \sum \vec{M}_O + \sum \vec{M}$$

Example: find an equivalent system of forces acting on O.



Section 4.8 Further Simplification of a Force and Couple System

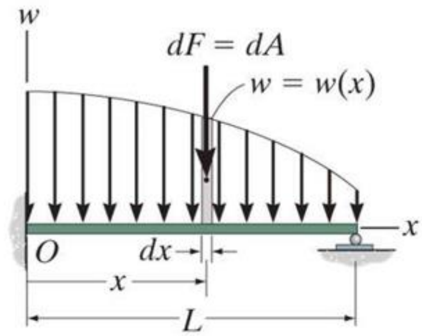


If $\vec{F}_R$ and $(\vec{M}_R)_O$ are perpendicular to each other, they can be further simplified by placing F_R at a distance d from the moment, which is found by:

$$d = \frac{(\vec{M}_R)_O}{\vec{F}_R}$$

Try it yourself: Problem F4-32

Section 4.9 Reduction of a Simple Distributed Loading



Consider a one-dimensional beam subjected to a distributed load, as shown. If we take an infinitesimally small element of the load, the force acting on that element is:

$$dF = w\,dx = dA$$

Now, summing the infinitesimally small elements, we can find the total force acting on the beam by:

$$F_R = \int dF = \int w\,dx = \int dA = A$$

So, the net force acting on the beam is equal to the area of the distributed load.

Additionally, the force will also produce a moment about point O. The moment caused by an infinitesimally small element of force is:

$$dM = x\,dF$$

Integrating, we can find the total moment acting on the beam:

$$M_R = \int dM = \int x\,dF = \int x\,w\,dx$$

We can set $\bar{x}$ as the distance where the resultant force acts on, so that our equation reduces to:

$$M_R = \bar{x} \int w\,dx$$

Now, combining both equations gives us:

$$\bar{x} = \frac{\int x\,w\,dx}{\int w\,dx} = \frac{\int x\,dA}{\int dA}$$

This is known as the centroid of the force distribution.

For now, we will focus on rectangular and triangular loads (so that their areas and centroids are easy to calculate).

Try it yourself: Problem F4-38